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NUMERICAL STUDY OF FREE OSCILLATIONS OF A BEAM WITH OSCILLATORS

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Free oscillations of a free-ended beam of variable cross section and mass with spring-suspended point masses (oscillators) are considered. It is found that parametric resonances are possible in this oscillating system. The effectiveness of the proposed calculation procedure is confirmed numerical calculations.

Key words: *free oscillations, beam of variable cross section and masses, oscillators.*

Introduction. Eigenvalue problems in which the solutions of the corresponding equations are smooth functions are considered in [1]. However, some problems of mathematical physics lead to eigenvalue problems with piecewise smooth functions. In the present paper, the results of [1] are extended to problems with piecewise smooth functions. An estimate of the error of the proposed method is given in [2]. The corresponding programs in Fortran are given in [3].

1. Formulation of the Problem. We consider a beam ($0 \leqslant x \leqslant L$) of cross section $F(x)$ (which is, generally speaking, variable) with masses m_k (oscillators) suspended by springs with rigidity c_k. It is required to determine the free oscillations of this system.

We consider the steady-state oscillations of one load (oscillator). The equation of motion of the load (oscillator) is written as

$$m\,\frac{d^2 z(t)}{dt^2} = -c(z(t) - u(x^*, t)),$$

where m is the mass of the load, c is the spring rigidity, $z(t)$ is the amplitude of the oscillator, and $u(x^*, t)$ is the displacement of the point of suspension $x = x^*$ of the oscillator. We set

$$z(t) = \mathrm{e}^{i\omega t}\, z, \qquad u(x, t) = \mathrm{e}^{i\omega t}\, u(x).$$

Then,

$$(u(x^*) - z)c + m\lambda z = 0, \qquad \lambda = \omega^2.$$

The motion of the beam is described by the equation

$$m\,\frac{\partial^2 u}{\partial x^2} = \frac{\partial \sigma}{\partial x} + \sum F,$$

where σ is the stress in the beam. From this, we obtain

$$\frac{d}{dx}\, EF(x)\, \frac{d}{dx}\, u(x) + \lambda m(x) u(x) - \sum_{k=1}^{n} c_k (u_k - z_k)\delta(x - x_k) = 0; \tag{1.1}$$

$$c_k(u_k - z_k) + \lambda m_k z_k = 0, \qquad \lambda_k = c_k / m_k; \tag{1.2}$$

$$EF(x)\, \frac{du}{dx}\Big|_{x=0,L} = 0. \tag{1.3}$$

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Boundary conditions (1.3) imply that the ends of the beam are free. Let us introduce into consideration the Green function $U = U(x, y)$ as a solution of the following problem:

$$\frac{d}{dx} EF(x) \frac{d}{dx} U(x, y) + \delta(x - y) - \frac{m(x)}{M} = 0, \qquad 0 < y < L; \tag{1.4}$$

$$EF(x) \frac{dU(x, y)}{dx}\Big|_{x=0,L} = 0; \tag{1.5}$$

$$\int_0^L m(x) U(x, y)\, dx = 0. \tag{1.6}$$

We note that Eq. (1.1) can be considered as a static one, i.e., as the problem of the extension of a beam of variable cross section under the action of the bulk force $[-\lambda m(x)u(x)]$ and point forces. By virtue of this, we have two elastic systems (1.1)–(1.3) and (1.4)–(1.6) according to the Betti reciprocal theorem: the work of the forces of the first system in displacements of the second system is equal to the work of the forces of the second system in displacements of the first system. As a result, we have

$$u(x) + \frac{1}{M} \sum_{k=1}^{n} m_k z_k = \lambda \int_0^L U(x, \xi) m(\xi) u(\xi)\, d\xi + \lambda \sum_{k=1}^{n} m_k z_k U(x, x_k); \tag{1.7}$$

$$\lambda_k (u_k - z_k) + \lambda z_k = 0, \qquad \lambda_k = c_k/m_k \tag{1.8}$$

[$u_k = u(x_k)$, where $k = 1, 2, \ldots, n$ is the value of the displacement at the attachment point of the kth oscillator].

Thus, we have the system of homogeneous integroalgebraic equations (1.7) and (1.8) for the free-oscillation amplitudes $u(x)$, $z_1, z_2, \ldots, z_n$, and frequency λ. For the case of forced oscillations, we have an inhomogeneous integral equation.

2. Quadrature Formula. The integral equation (1.7) is discretized using the quadrature formula

$$c_k = \int_{x_{k-1}}^{x_{k+1}} m(x) S_k(x)\, dx, \tag{2.1}$$

where

$$S_k(x) = \begin{cases} \dfrac{x - x_{k-1}}{x_k - x_{k-1}}, & x \in [x_{k-1}, x_k], \\[2ex] \dfrac{x - x_{k+1}}{x_k - x_{k-1}}, & x \in [x_k, x_{k+1}], \end{cases} \qquad k = 2, \ldots, N-1;$$

$$S_1(x) = \begin{cases} 1, & x \in [x_0, x_1], \\ \dfrac{x - x_2}{x_1 - x_2}, & x \in [x_1, x_2]; \end{cases} \qquad S_N(x) = \begin{cases} \dfrac{x - x_{N-1}}{x_N - x_{N-1}}, & x \in [x_{N-1}, x_N], \\ 1, & x \in [x_N, L]; \end{cases}$$

$m(x)$ is a piecewise linear function whose discontinuity (break) points coincide with the mesh nodes. Thus, the function $m(x)$ is linear on each segment. On the ith segment, let $m(x) = a_i x + b_i$, where a_i and b_i are certain constants.

To simplify the programming procedure, we use the Simpson formula

$$\int_{x_0}^{x_2} f(x)\, dx = \frac{h}{3} [f_0 + 4f_1 + f_2].$$

We denote $x_k^* = (x_k + x_{k-1})/2$ and $x_{k+1}^* = (x_{k+1} + x_k)/2$. Then, for $k = 1, 2, \ldots, N$, we have

$$c_k = \int\limits_{x_{k-1}}^{x_k} m(x) S_k(x)\, dx + \int\limits_{x_k}^{x_{k+1}} m(x) S_k(x)\, dx$$

$$= \frac{x_k - x_{k-1}}{6} \left[m(x_{k-1}) S_k(x_{k-1}) + 4m(x_k^*) S_k(x_k^*) + m(x_k) S_k(x_k) \right]$$

$$+ \frac{x_{k+1} - x_k}{6} \left[m(x_k) S_k(x_k) + 4m(x_{k+1}^*) S_k(x_{k+1}^*) + m(x_{k+1}) S_k(x_{k+1}) \right].$$

Here

$$S_k(x_{k-1}) = \begin{cases} 0, & k \neq N, \\ 1, & k = N; \end{cases} \qquad S_k(x_{k+1}) = \begin{cases} 0, & k \neq N, \\ 1, & k = N; \end{cases} \qquad S_k(x_k) = 1.$$

3. Discretization. Calculating the integral term in (1.7) by the quadrature formula (2.1), we obtain a finite-dimensional problem of the form

$$\begin{pmatrix} 1 & \dots & 0 & m_1/M & \dots & m_n/M \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & m_1/M & \dots & m_n/M \\ & & & \lambda_1 & \dots & 0 \\ & J & & \vdots & \ddots & \vdots \\ & & & 0 & \dots & \lambda_n \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_N \\ z_1 \\ \vdots \\ z_n \end{pmatrix} = \lambda \begin{pmatrix} A & U\hat{M} \\ 0 & I_n \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_N \\ z_1 \\ \vdots \\ z_n \end{pmatrix}$$

or in the matrix form

$$E \begin{pmatrix} u \\ z \end{pmatrix} = \lambda D \begin{pmatrix} u \\ z \end{pmatrix}. \tag{3.1}$$

Here

$$A_k(x) = \int\limits_{x_{k-1}}^{x_{k+1}} U(x,\xi) m(\xi) S_k(\xi)\, d\xi, \qquad k = 1, 2, \dots, N;$$

$$A_{ik} = A_k(x_i), \qquad k, i = 1, 2, \dots, N;$$

$$\int\limits_0^L U(x,\xi) m(\xi) u(\xi)\, d\xi = \sum_{k=1}^N c_k A_k(x) u_k$$

[by setting $u(\xi) = \sum\limits_{k=1}^N u_k S_k(\xi)$ and $u_k = u(x_k)$], J is $n \times N$ matrix in which $j(k)$ in the kth line ($k = 1, 2, \dots, n$) is replaced by $-\lambda_k$ and the remaining terms in this line are zero [$j(k)$ is an integer function that associates the oscillator number k to the mesh node number], U is a matrix of size $N \times n$, $U_{ik} = U(x_i, x_{j(k)})$, and $\hat{M} = \mathrm{diag}\,(m_1, m_2, \dots, m_n)$ is a diagonal matrix of size $n \times n$.

In relation (3.1), the matrix E can be inverted analytically, resulting in a finite-dimensional problem of the form

$$v = \lambda E^{-1} D v, \qquad v = (u, z),$$

where

$$E^{-1} = \begin{pmatrix} I_N + m(\Lambda - Jm)^{-1} J & -m(\Lambda - Jm)^{-1} \\ -(\Lambda - Jm)^{-1} J & (\Lambda - Jm)^{-1} \end{pmatrix};$$

$$m = \frac{1}{M} \begin{pmatrix} m_1 & \dots & m_n \\ \vdots & \ddots & \vdots \\ m_1 & \dots & m_n \end{pmatrix}; \qquad \Lambda = \mathrm{diag}\,(\lambda_1, \dots, \lambda_n)$$

(m is a matrix of size $N \times n$);

$$E^{-1}D = \begin{pmatrix} (I_N + m\hat{\Lambda}J)A & (I_N + m\hat{\Lambda}J)U\hat{M} - m\hat{\Lambda} \\ -\hat{\Lambda}JA & -\hat{\Lambda}JU\hat{M} + \hat{\Lambda} \end{pmatrix}. \tag{3.2}$$

We note that $Jm = -\Lambda\hat{m}$ and, hence, $\hat{\Lambda} = (I_n + \hat{m})^{-1}\Lambda^{-1}$, where

$$\hat{m} = \frac{1}{M}\begin{pmatrix} m_1 & \dots & m_n \\ \vdots & \ddots & \vdots \\ m_1 & \dots & m_n \end{pmatrix}$$

is a matrix of size $n \times n$. The matrix $I_n + \hat{m}$ is easily inverted analytically. We denote

$$p_i = \Big(1 + \frac{1}{M}\sum_{i\neq j} m_j\Big) \Big/ \Big(1 + \frac{1}{M}\sum_{j} m_j\Big), \qquad i = 1, 2, \dots, n,$$

$$(I_n + \hat{m})^{-1} = \begin{pmatrix} p_1 & p_2 - 1 & \dots & p_n - 1 \\ p_1 - 1 & p_2 & \dots & p_n - 1 \\ \vdots & \vdots & \ddots & \vdots \\ p_1 - 1 & p_2 - 1 & \dots & p_n \end{pmatrix} = P.$$

Then,

$$\hat{\Lambda} = \begin{pmatrix} p_1 & p_2 - 1 & \dots & p_n - 1 \\ p_1 - 1 & p_2 & \dots & p_n - 1 \\ \vdots & \vdots & \ddots & \vdots \\ p_1 - 1 & p_2 - 1 & \dots & p_n \end{pmatrix} \begin{pmatrix} \lambda_1^{-1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n^{-1} \end{pmatrix}.$$

For convenience of programming, we write Eq. (3.2) as

$$E^{-1}D = \begin{pmatrix} A + m\hat{\Lambda}JA & U\hat{M} + m\hat{\Lambda}(JU\hat{M} - I) \\ -\hat{\Lambda}JA & \hat{\Lambda} - \hat{\Lambda}JU\hat{M} \end{pmatrix}.$$

4. Examples of Numerical Calculations. Numerical calculation were performed for a homogeneous beam $EF(x) = 1$ and $m(x) = 1$ of unit length. A uniform mesh of 68 internal nodes was used. Oscillators with masses of 0.1 and frequencies of 5 and 6 were attached at the 18th and 36th nodes to the beam. The first five eigenfrequencies were obtained: 4.7240, 6.4569, 10.7744, 40.8333, and 89.3858. Figures 1–5 show the eigenmodes corresponding to these frequencies.

The problem described above was solved for both constant distributions $EF(x) = \text{const}$ and $m(x) = \text{const}$ and for piecewise linear distributions, and the functions $EF(x)$ and $m(x)$ had several discontinuities. To control the procedure for $EF(x) = 1$ and $m(x) = 1$, we reduced the problem to the general eigenvalue problem for a meromorphic λ-matrix $A(\lambda)$. A comparison of the calculations using these procedures shows that the spectrum of the problem consists of two parts: the perturbed frequencies of the oscillators and the perturbed frequencies of the beam without oscillators. The first group of frequencies is determined almost exactly. For example, the minimum eigenvalue in the system with 14 oscillators is determined with an absolute error of 10^{-8}. The amplitudes of the oscillators calculated by the two procedures also coincide with high accuracy. The second group of frequencies is calculated with an error $O(h)$.

The calculations for the general case were performed for piecewise linear functions $EF(x)$ and $m(x)$ that had a few tens of breaks and a few discontinuities. The results were compared with calculations using the shooting method. In the cases where it was possible to calculate the eigenvalues using the shooting method, the results coincided qualitatively with the results obtained for a homogeneous beam.

In particular calculations, the equations written above were rendered dimensionless. As the characteristic mass and lengths, we used the mass of the beam without oscillators and the length of the beam, and as the characteristic time, the quantity $1/W_{\max}$, where $W_{\max}$ is the characteristic frequency (the end of the computation range). The calculations were performed both for a methodical purpose and for the purpose of studying the occurrence of parametric resonance in the complex oscillating system.

Calculations were also performed for a beam with 14 oscillators. In one calculation, eight of them had identical frequency λ_0 and one (second) oscillator had a frequency close to λ_0. The remaining oscillators had

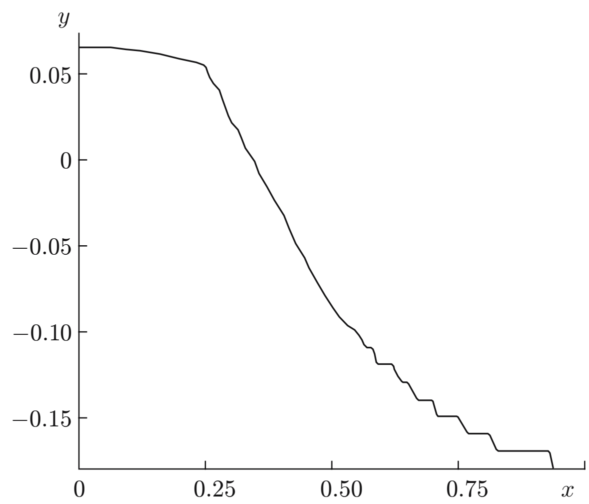

Fig. 1

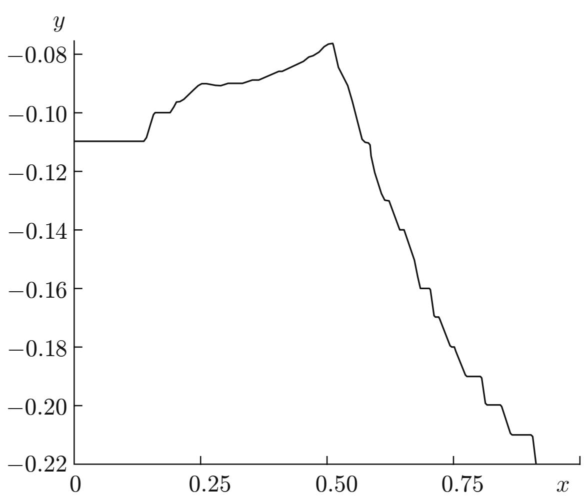

Fig. 2

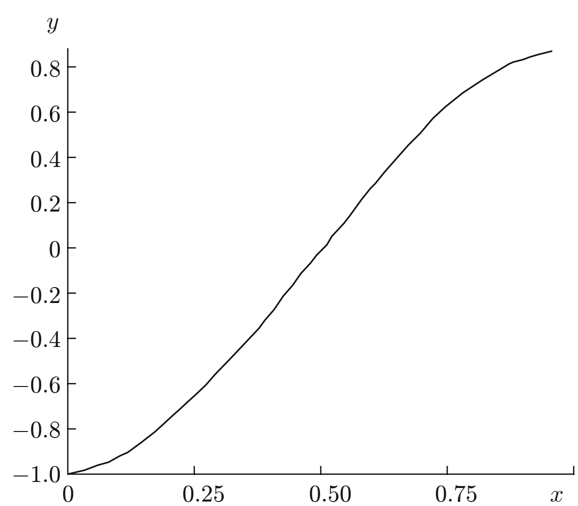

Fig. 3

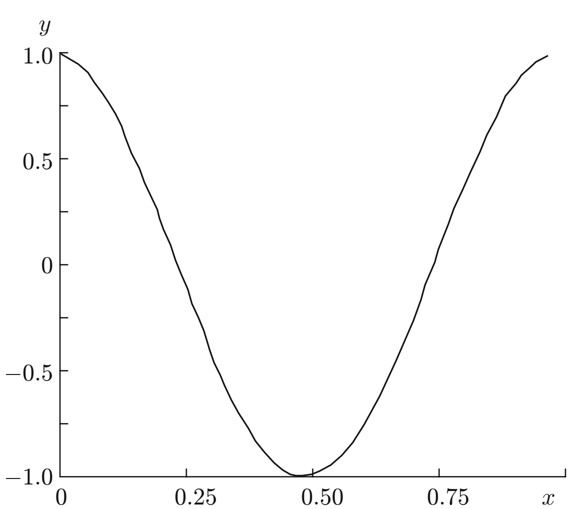

Fig. 4

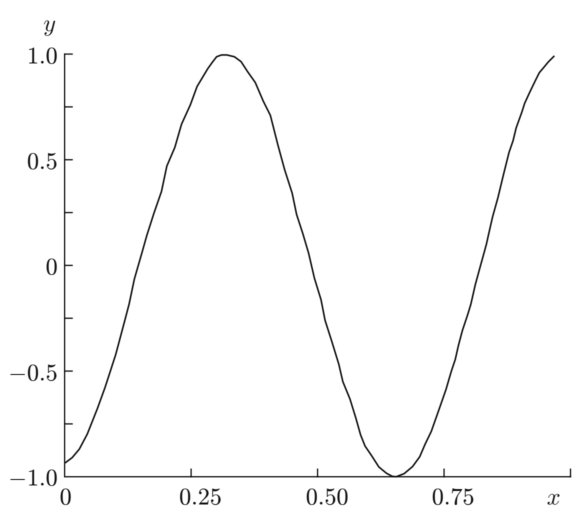

Fig. 5

different frequencies. The distance between one pair of oscillators with frequency λ_0 (the 5th and 6th oscillators) was $3 \cdot 10^{-3}\%$ of the length of the beam, and that between the other pair (the 12th and 13th oscillators) was 4.5% of the beam length.

Four calculations were performed: 1) $EF(x) \equiv 1$ and $m(x) \equiv 1$; 2) $EF(x) \equiv 1$ and $m(x) \neq 1$; 3) $EF(x) \neq 1$ and $m(x) \neq 1$; 4) $EF(x) \neq 1$ and $m(x) \equiv 1$. In all calculations, the functions $EF(x)$ and $m(x)$ had identical form (if they were not identically equal to unity), the same mesh was used, and the masses, frequencies, and arrangement of the oscillators was unchanged.

It turned out that the oscillating system had a frequency close to but slightly smaller than λ_0. For versions 1 and 2, the difference was $2.8 \cdot 10^{-3}\%$, and for versions 3 and 4, it was $1.3 \cdot 10^{-3}\%$. The 6th oscillator had the maximum amplitude, and the amplitude of the 5th oscillator was approximately 60% of the amplitude of the 6th oscillator. The remaining oscillators had the following amplitudes: 0.1% (1, 2, and 7th), $0.1 \cdot 10^{-3}\%$ (3rd and 4th), and 10^{-5}–$10^{-13}\%$ (8–14th) of the amplitude of the 6th oscillator. The points of suspensions of the oscillators had amplitudes not larger than 0.01% of the amplitude of the 6th oscillator.

Thus, all oscillators, except for the 5th and 6th, are almost stationary, the moving oscillators (the 5th and 6th) oscillate in antiphase (their amplitudes have opposite signs), and the following approximate equality is satisfied:

$$\frac{\text{amplitude of 6th oscillator}}{\text{amplitude of 5th oscillator}} = -\frac{\text{mass of 5th oscillator}}{\text{mass of 6th oscillator}}.$$

From these equalities it follows that the forces acting on the beam at the attachment points of the 5th and 6th oscillators are opposing and are almost equal in magnitude (in the calculation, the difference was 1.5%). Compared to these forces, the forces exerted on the beam by the remaining oscillators amount to not more than 0.01%.

For version 1, a different calculation was performed. The distance between the points of suspension of the thirteenth and twelfth oscillators was $2.6 \cdot 10^{-3}\%$ of the beam length. In this case, the system has two frequencies close to λ_0 (both smaller than λ_0). At one of these frequencies (which differs from λ_0 by approximately $2.8 \cdot 10^{-3}\%$) the 5th and 6th oscillators have the larger amplitude, and at the other (which differs from λ_0 by approximately $4.1 \cdot 10^{-4}\%$), the larger amplitude is observed for the 12th and 13th oscillators. The oscillation pattern is qualitatively similar to that described above but is more pronounced.

For version 3, a calculation was performed with a changed frequency of the 2nd oscillator (the frequency was increased by a factor of 100). The frequency of the system close to λ_0 remained almost unchanged. The amplitudes of the oscillators (for oscillation of the system with a frequency close to λ_0) increased (except for the amplitudes of the 5th and 6th oscillators) but the pattern is qualitatively similar to that described above.

In the last calculation for version 3, the 6th and 5th oscillators were separated by a distance equal to 1% of the beam length. The effect described above disappeared, and the frequency of the oscillating system the nearest to λ_0 was 6% of the value of λ_0. The calculation results for the version with 14 oscillators are given in [2].

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